Accurate Resonant Frequencies of Dielectric Resonators

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Abstract—**The applications of dielectric resonators are currently of considerable interest in microwave integrated circuits [1]–[4].**

The design of a dielectric resonator depends on its natural resonant frequencies. Since exact solutions of dielectric rectangular and cylindrical resonators cannot be rigorously computed a new and more accurate method has been developed to solve this problem. In this method all the surfaces are simultaneously considered as imperfect magnetic walls.

Theoretical values agree very well with experimental results. The difference being less than 1 percent.

I. Introduction

Dielectric resonators made from low-loss high-permittivity dielectric material offer the possibility for miniaturization of high-Q microwave filters [5]. Bell Laboratories has recently developed a consistent processing technique for fabricating titanate ceramic having the desired loss tangent in the microwave frequency region [6].

The resonant frequency of rectangular and cylindrical dielectric resonators has been traditionally analyzed by using the magnetic wall model [5], [7]–[10]. This method gives a difference of about 10 percent between theoretical and experimental results.

A more accurate method based on the variation method has been reported by Konishi [11]; theoretical and experimental results agree with an error less than 1 percent.

Recently Itoh and Rudokas [12] reported a numerical procedure based on this analysis of the propagation characteristics of the three-dimensional cylindrical resonator structure. Although this method was less accurate than that of Konishi it provides a good agreement between experimental and theoretical results.

In this paper a new procedure is reported for predicting the resonant frequencies of lower and higher modes of isolated and shielded resonators of rectangular and cylindrical shapes.

II. Method of Analysis

Let us consider a homogeneous, lossless circular dielectric resonator: relative permittivity ε_2, radius a, height H.

Four approximations may be defined to evaluate the resonant frequencies of dielectric resonators.

Approximation 1: Assume that all the surfaces are perfect magnetic walls:

$$\vec{n} \times \vec{H} = 0$$

$$\vec{n} \cdot \vec{E} = 0$$

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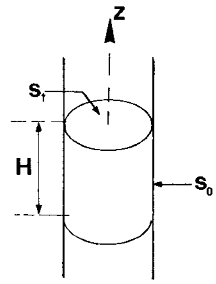

Fig. 1. The isolated cylindrical resonator with its lateral surface, which is a perfect magnetic wall.

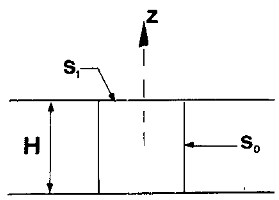

Fig. 2. The isolated cylindrical resonator with its flat surfaces, which are perfect magnetic walls.

where $\vec{n}$ is the unit vector normal to the boundary and $\vec{E}$ and $\vec{H}$ are, respectively, the electric and magnetic field vectors.

Approximation 2_H*:* Assume that the lateral surface S_0 (Fig. 1) is a magnetic wall and S_1 is an imperfect magnetic wall.

Approximation 2_a*:* S_1 is a perfect magnetic wall and S_0 is an imperfect magnetic wall (Fig. 2).

Approximation 3: All the surfaces (S_0,S_1) of the dielectric resonator are simultaneously imperfect magnetic walls.

To evaluate the resonant frequency with such approximations, we have to solve a system of two coupled characteristic equations: one is obtained by assuming that only the lateral surface (S_1) is a perfect magnetic wall (approximation 2_a); the other, by assuming that only the flat surfaces (S_0) are perfect magnetic wall (approximation 2_H).

The method used to improve the accuracy of the resonant frequency consists of two steps.

1) Successively apply these approximations in order to determine an effective dielectric resonator: radius a_e, height H_e, relative permittivity ε_2. Then apply approximation 1 to obtain the resonant frequency f_e of the effective resonator.

2) Apply approximation 3 to the real resonator (a,H,ε_2) to obtain its resonant frequency f_i.

The accurate theoretical resonant frequency f of the dielectric resonator (a,H,ε_2) is obtained by taking the mean value of f_e and f_i.

To define the effective resonator, we proceed as follows.

Effective Radius: We solve approximation 2_H to obtain the resonant frequency of the (a,H) resonator. Let f_{1H} be this frequency. Using approximation 1 to obtain a resonator which has an f_{1H} resonant frequency and a radius a this resonator should have a height H_1.

With approximation 2_a, the resonant frequency of an (a,H_1) resonator is f_{1a}.

The radius of a resonator of height H_1 resonating at f_{1a} is a_e in approximation 1.

Effective Height: We solve the eigenvalue equation 2_a to obtain the resonant frequency f_{2a} of the dielectric resonator (a,H).

Now assuming the first-order approximation 1 we may determine the radius a_1 of the resonator of height H which resonates at f_{2a}. Considering the new resonator $(a_1 H)$ we apply approximation 2_H to determine its resonant frequency. Let f_{2H} be this frequency. The first-order approximation gives the effective height H_e of a resonator of radius a_1 and of resonant frequency f_{2H}.

Frequency of the Effective Resonator: f_e is obtained by assuming that the walls of the effective resonator are perfect magnetic ones. The resonant frequency obtained with such a consideration is f_e.

To outline the method we worked out a table which is available for all the resonant modes of a dielectric disk resonator.

Determination of a_e:

$$(a,H) + 2_H \rightarrow f_{1H}$$
$$(f_{1H},a) + 1 \rightarrow H_1$$
$$(a,H_1) + 2_a \rightarrow f_{1a}$$
$$(f_{1a},H_1) + 1 \rightarrow a_e.$$

Determination of H_e:

$$(a,H) + 2_a \rightarrow f_{2a}$$
$$(f_{2a},H) + 1 \rightarrow a_1$$
$$(a_1,H) + 2_H \rightarrow f_{2H}$$
$$(f_{2H},a_1) + 1 \rightarrow H_e.$$

Determination of f_e:

$$(a_e H_e) + (1) \rightarrow f_e.$$

Determination of f_i:

$$(a,H) + (3) \rightarrow f_i.$$

Determination of the Frequency f of the Real Resonator:

$$f = \tfrac{1}{2}(f_e + f_i).$$

This method outlined for a cylindrical resonator is still valid for analyzing the resonant frequencies of rectangular resonators. We shall study the eigenvalue equations which are necessary to determine the resonant frequencies of the modes of cylindrical and rectangular resonators.

III. The Cylindrical Resonator

A. The Isolated Cylindrical Resonator (a,H,ε_2)

1) *Definition of Approximation 2_H:* We write the magnetic and electric longitudinal components (Appendix). Substituting these expressions into Maxwell equations and matching the solutions at $z = \pm H/2$, we obtain the characteristic equation which gives the resonant frequencies:

$$H = \frac{2}{\beta}\tan^{-1}\frac{\alpha}{\beta} \tag{1}$$

$$k_o = \omega\sqrt{\varepsilon_o\mu_o} \qquad \alpha^2 = k_c^2 - k_o^2 \qquad \beta^2 = k_o^2\varepsilon_2 - k_c^2 \tag{2}$$

where k_c is given by

$$J_n(k_c a) = 0, \qquad \text{for } \mathrm{TE}_{n,m,p} \text{ modes} \tag{3}$$

$$J_n'(k_c a) = 0, \qquad \text{for } \mathrm{TM}_{n,m,p} \text{ modes} \tag{4}$$

and $J_n(k_c a)$ is the nth order Bessel's function of the first kind.

2) *Definition of Approximation 2_a:* We assume that the flat surfaces (S_1) satisfy the open-circuit boundary conditions, but the cylindrical surface does not.

a) *Hybrid $EH_{n,m,p}$ modes:* All the modes, except those with cylindrical symmetry are hybrid EH modes.

The expressions for E and H longitudinal fields components are given in the Appendix. Using these relations, (20) and (21), and Maxwell's equations, we have a complete set of field expressions.

Matching the electric and magnetic fields components at $r = a$, gives the secular equation [13]

$$\left\{\frac{\varepsilon_2 J_n'(\rho_i)}{\rho_i J_n(\rho_i)} - \frac{K_n'(\rho_o)}{\rho_o \cdot K_n(\rho_o)}\right\}\cdot\left\{\frac{J_n'(\rho_i)}{\rho_i J_n(\rho_i)} + \frac{K_n'(\rho_o)}{\rho_o K_n(\rho_o)}\right\} = \frac{n^2\beta'^2}{k_o^2}\cdot\left\{\frac{k_{c_o}^2 + k_{c_i}^2}{k_{c_o}^2 k_{c_i}^2}\right\}^2 \tag{5}$$

where

$$\rho_i = k_{c_i}a \qquad \rho_o = k_{c_o}a \tag{6}$$

$$k_{c_i}^2 = k_o^2\varepsilon_2 - \beta'^2 \qquad k_{c_o}^2 = \beta'^2 - k_o^2, \qquad \beta' = \frac{p\pi}{H} \tag{7}$$

i designates the mode dielectric, and o designates the outside dielectric.

b) $\mathrm{TE}_{o,m,p}$ *and* $\mathrm{TM}_{o,m,p}$ *modes:* For the case of $\mathrm{TE}_{o,m,p}$ and $\mathrm{TM}_{o,m,p}$ modes, relation (5) reduces to

$$\rho_i\frac{J_o(\rho_i)}{J_o'(\rho_i)} = -\frac{\varepsilon_1}{\varepsilon_2}\rho_o\frac{K_o(\rho_o)}{K_o'(\rho_o)} \tag{8}$$

for $\mathrm{TM}_{o,m,p}$ modes

$$\rho_i\frac{J_o(\rho_i)}{J_o'(\rho_i)} = -\frac{\mu_1}{\mu_2}\rho_o\frac{K_o(\rho_o)}{K_o'(\rho_o)} \tag{9}$$

for $\mathrm{TE}_{o,m,p}$ modes.

3) *Definition of Approximation 3:* To explain the principle of this approximation we consider the case of the dipolar $\mathrm{TE}_{o,m,p}$ mode.

We consider that all the surfaces (cylindrical and flat surfaces) of the dielectric resonator (a,H,ε_2) are simultaneously imperfect magnetic walls. So, in order to obtain the resonant frequency, we have to solve the coupled equations

$$\rho_i\frac{J_o(\rho_i)}{J_o'(\rho_i)} = -\rho_o\frac{K_o(\rho_o)}{K_o'(\rho_o)} \tag{10}$$

$$\left(\frac{H\beta}{2}\right)^2\left(1 + \tan^2\frac{H\beta}{2}\right) = \left(\frac{2\pi f}{c}\right)^2(\varepsilon_2 - 1). \tag{11}$$

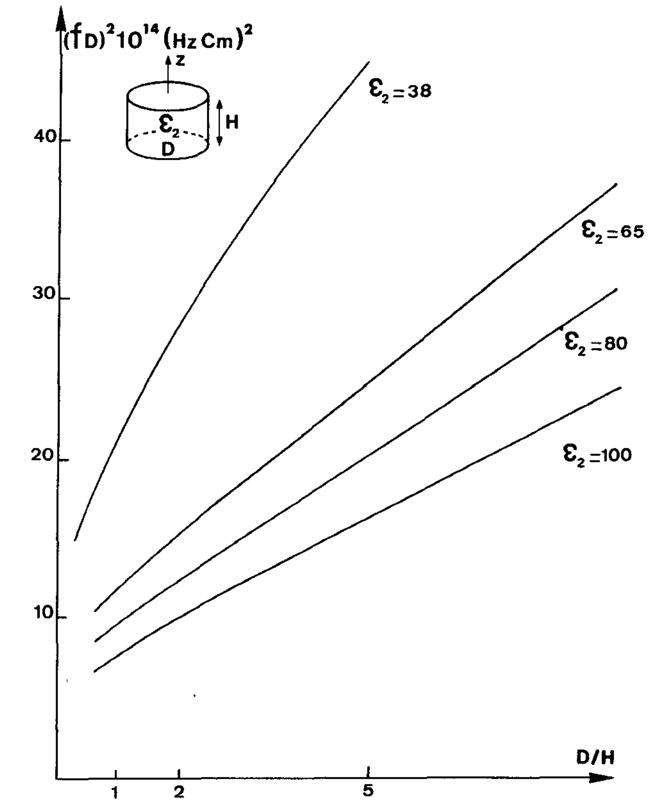

Fig. 3. Resonant frequencies of the TE_{01p} mode of isolated cylindrical resonators.

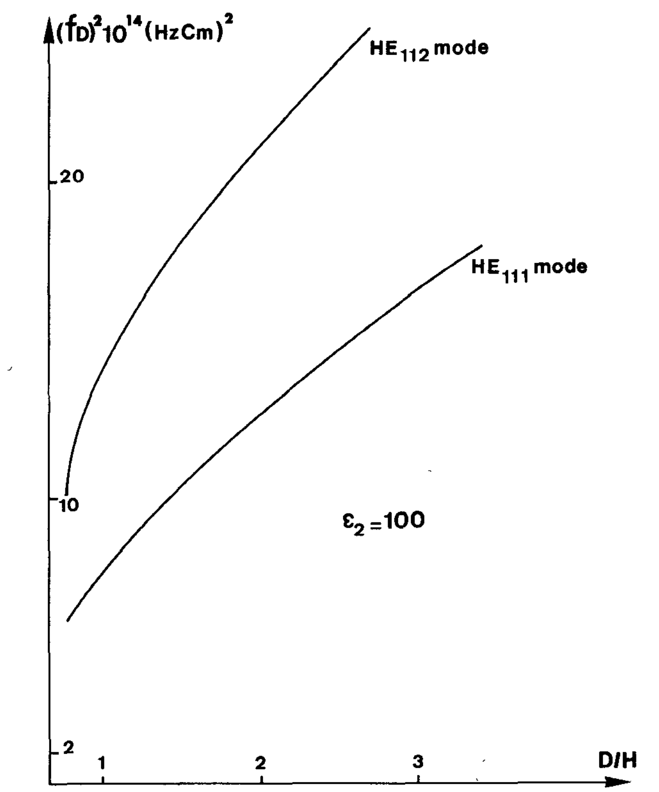

Fig. 4. Resonant frequencies of the HE_{11p} and $HE_{11\,2p}$ modes of an isolated cylindrical resonator for $\varepsilon_2 = 100$.

Equation (11) is obtained by substituting α's value into (1).

4) *Experimental Results:* Using approximations 1, 2_a, 2_H, and 3, we have realized a computer program which permits us, by applying the method described in (7), to obtain the resonant frequencies as a function of the parameters of the dielectric resonator. The results obtained for an isolated resonator are given in Figs. 3 and 4.

B. The Shielded Cylindrical Resonator

1) *Definition of Approximation 2_H for a Resonator Placed in a Microstrip Structure (Fig. 5):* The effect of the metallic walls is to modify the exponential decay of field components outside the resonator. The tangential components of an electric field must vanish on perfectly conducting surfaces.

Matching tangential components of electric and magnetic fields at the boundaries results in a set of linear equations. The characteristic equation which gives the resonant frequencies of $TE_{n,m,p}$ modes is then

$$\left(\beta \tan \beta\frac{H}{2}\cdot \tanh \alpha_3 H - \alpha_3\right)\cdot\left(\alpha_1 \tan \beta\frac{H}{2} + \beta \tanh \beta d\right) + \left(\beta \tan \beta\frac{H}{2}\cdot \tanh \alpha_1 \cdot d - \alpha_1\right) \cdot \left(\alpha_3 \tan \beta\frac{H}{2} + \beta \tanh \alpha_3 h\right) = 0 \quad (12)$$

where

$$\alpha_3^2 = k_c^2 - k_o^2\varepsilon_3$$
$$\alpha_1^2 = k_c^2 - k_o^2\varepsilon_1.$$

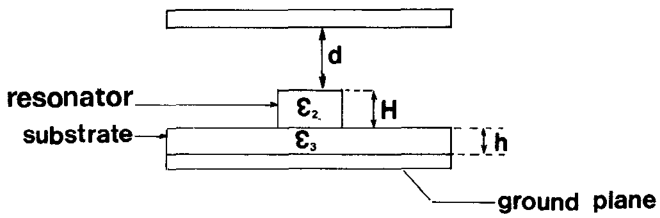

Fig. 5. The shielded cylindrical dielectric resonator.

The method described in Section II is applied but now we use (12) and (9) to calculate f_e and f_i and thus f of the $TE_{n,m,p}$ modes. Theoretical results are presented in Figs. 6 and 7.

Fig. 6 indicates a very important property of the resonant frequency behavior of the two lowest order modes with D/H. The HE_{11p} is the lowest frequency mode for $(D/H) \leq 1.42$. Above $(D/H) = 1.42$, the TE_{01p} mode becomes the lowest frequency resonance. In the vicinity of $(D/H) = 1.42$, where the modes overlap, it is possible to use a single dielectric resonator as a dual-mode resonator in order to do two-pole bandpass filtering for wider bandwidths.

IV. The Rectangular Resonator

A. The Isolated Rectangular Resonator

The method outlined above is also available to determine the resonant frequencies of a rectangular resonator.

1) *Definition of Approximation 2_H*

a) *$TE_{n,m,p}$ modes:* To determine approximation 2_H we assume that the a and b surfaces satisfy the open-circuit

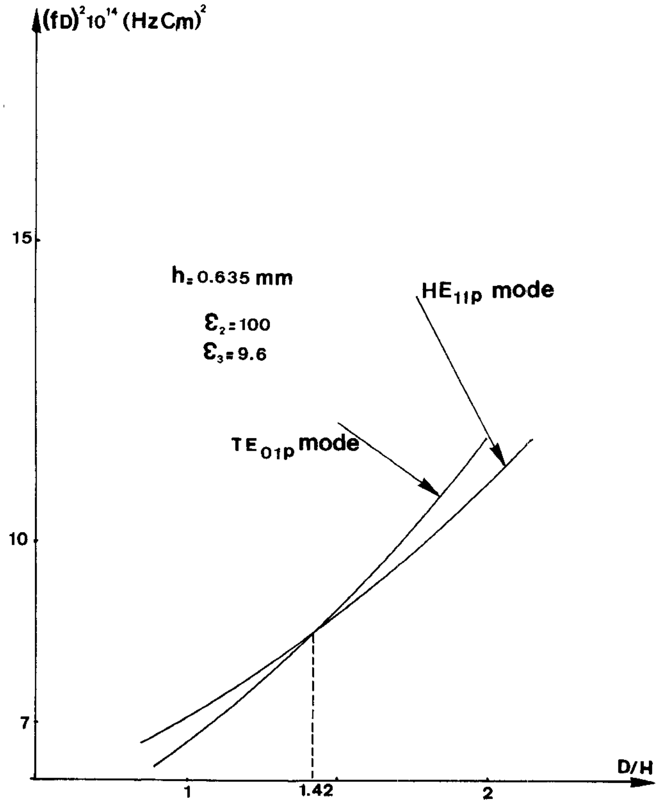

Fig. 6. Resonant frequencies of the TE_{01p} and HE_{11p} modes of a shielded cylindrical resonator for $\varepsilon_2 = 100$.

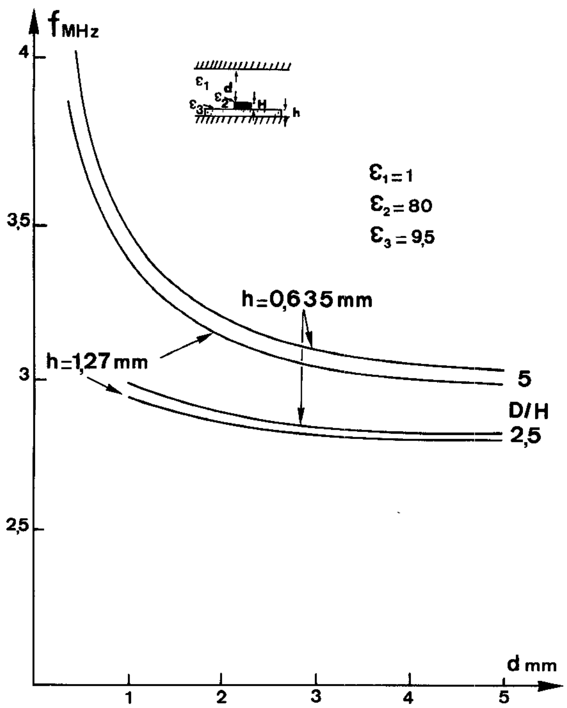

Fig. 7. Resonant frequencies of the TE_{01p} modes of a shielded cylindrical resonator for different thickness of the substrate.

boundary conditions (Fig. 8) while the fields decay exponentially along the surfaces perpendicular to the $0z$ propagation axis. The eigenvalue equation, (1), is still available to

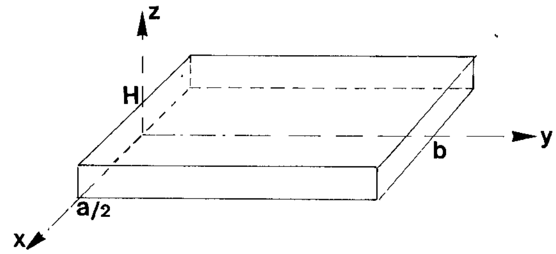

Fig. 8. The isolated rectangular resonator.

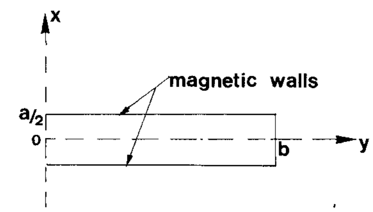

Fig. 9. The isolated rectangular resonator with two perfect magnetic walls.

determine the resonant frequencies of a dielectric rectangular resonator:

$$k_c^2 = \pi^2 \left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right). \tag{13}$$

b) *Hybrid modes:* More generally and for modes other than TE and TM modes, we can study the case for which two surfaces of the section satisfy the open-circuit boundary conditions.

As shown in Fig. 9, let $y = 0$, b and $z = 0$, and the H surfaces be perfect magnetic walls. The longitudinal E and H fields obtained with these conditions are given in the Appendix. Substituting these components' values into Maxwell equations and matching at $x = \pm a/2$, we obtain the characteristic equation

$$\beta' k_y^2 \left(\frac{1}{k_{c_1}^2} + \frac{1}{k_{c_2}^2}\right) + k_o^2 \varepsilon_2 \left(\frac{k_{x_2}^2}{k_{c_2}^4} + \frac{k_{x_1} k_{x_2}}{k_{c_1}^2 k_{c_2}^2} \tan k_{x_1} a/2\right) - k_o^2 \varepsilon_1 \left(\frac{k_{x_1}^2}{k_{c_1}^4} + \frac{k_{x_1} k_{x_2}}{k_{c_1}^2 k_{c_2}^2} \frac{1}{\tan k_{x_1} a/2}\right) = 0 \tag{14}$$

with

$$k_{x_2}^2 = k_o^2 \varepsilon_2 - (k_y^2 + \beta'^2) \qquad k_{x_1}^2 - (k_y^2 + \beta'^2) - k_o^2 \varepsilon_1,$$

$$k_y = \frac{m\pi}{b}.$$

This relation can be applied to the special case of LSE_x and LSM_x modes, respectively, with longitudinal electric and magnetic fields and with E_x and H_x equal to zero.

The characteristic equation (14) becomes

$$a = \frac{2}{k_{x_2}} \tan^{-1} \frac{k_{x_1}}{k_{x_2}} \tag{15}$$

for LSE_x modes

$$a = \frac{2}{k_{x_2}} \frac{1}{\tan^{-1} \frac{\varepsilon_2}{\varepsilon_1} \frac{k_{x_1}}{k_{x_2}}} \tag{16}$$

for LSM_x modes.

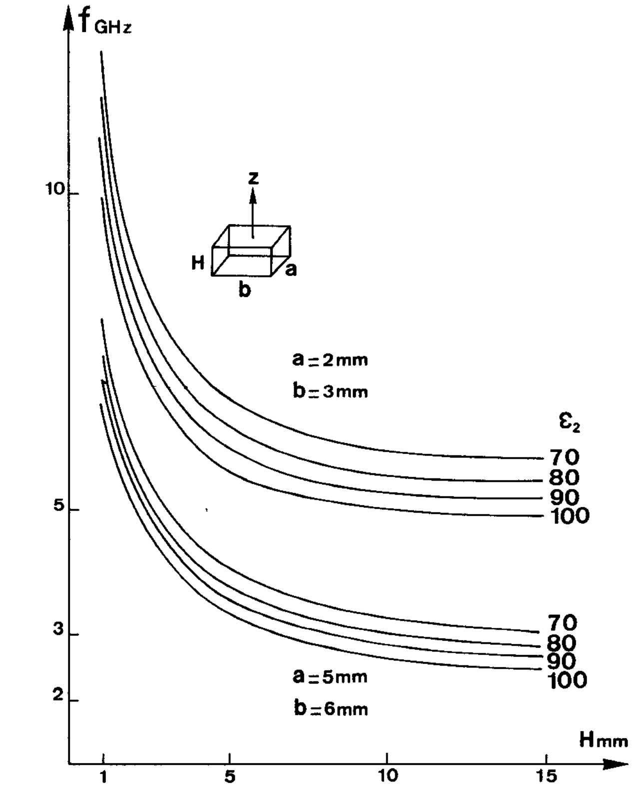

Fig. 10. Resonant frequencies of the TE_{11p} mode of an isolated rectangular resonator.

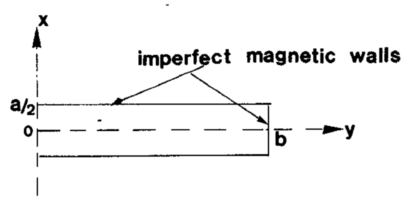

Fig. 11. The isolated rectangular resonator with two imperfect magnetic walls.

We have given in the Appendix the eigenvalue equation for hybrid modes when the surfaces $x = \pm a/2$ and $z = 0, H$ satisfy the open-circuit boundary conditions, (28).

Using the method described previously (Section II) we have evaluated the resonant frequencies of an isolated rectangular resonator (Fig. 10).

2) *Definition of Approximation* $2_{a,b}$: Only terminal surfaces ($z = 0$, $z = H$) (Fig. 11) of a rectangular resonator satisfy open-circuit boundary conditions.

We write the expressions of both the longitudinal and tangential components in each medium (Appendix), and, matching them simultaneously at the planes $x = o,a$ and $y = o,b$, obtain a determinant D (Appendix).

The characteristic equation which gives the resonant frequencies of hybrid modes of such a resonator is obtained from the requirement that D vanishes.

B. The Shielded Resonator

1) *Definition of Approximation* 2_H:

a) $TE_{n,m,p}$ *modes:* If the rectangular resonator is in a microstrip structure, relation (12) is also available but it is

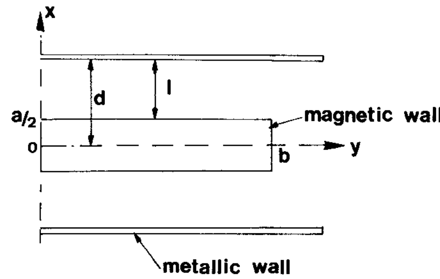

Fig. 12. The shielded rectangular resonator.

TABLE I

L mm	D mm	ε_2	F_e MHz	Experimental f f_{exp} MHz	$\frac{f_e - f_{exp}}{f_{exp}}$ %
7	10	38	4705	4660	1
8,4	15	38	3293	3315	0,7
7	15	38	3420	3450	1
4	10	38	5326	5372	0,9
5	15	66	2860	2850	0,4
7	10	66	3540	3506	1

now necessary to replace $(x_{n_m}/a)^2$ by $\pi^2((n^2/a^2) + (m^2/b^2))$ in the α_1 and α_3's values.

b) *Hybrid modes:* We consider the geometrical structure shown in Fig. 12 and determine the fields in each medium (Appendix). After having determined the tangential E and H components and matching them at the boundary, we obtain the characteristic equation necessary to calculate the resonant frequencies of the hybrid modes of a shielded rectangular resonator:

$$\varepsilon_2 \frac{k_{x_2}^2}{k_{c2}^4} + \varepsilon_1 \frac{k_{x_1}^2}{k_{c1}^4} + \frac{k_{x_1} k_{x_2}}{k_{c1}^2 k_{c2}^2} \cdot \left\{ \frac{\varepsilon_1}{\tan k_{x_2} a/2 \tan k_{x_1} l} - \varepsilon_2 \tan k_{x_2} a/2 \tan k_{x_1} l \right\} = 0. \quad (17)$$

V. Experimental Results

A. Isolated Resonator

We give results for the dipolar TE_{01p} mode of the cylindrical resonator in Table I.

A comparison shows that experimental and theoretical results agree within 1 percent.

Cohn has shown in his report [5] that to obtain good resonance frequencies by using approximation 2_H it is necessary to multiply the relative permittivity of the resonator ε_2 by 0.875. Our method confirms this value since we find a factor equal to 0.870.

B. Shielded Resonators

Resonator in a Microstrip Box: In Fig. 13 we have

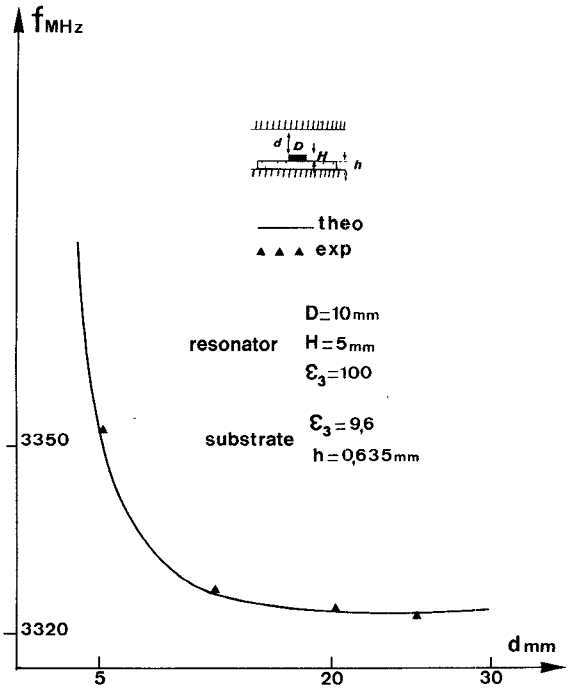

Fig. 13. Experimental resonant frequencies of the TE_{01p} modes of a shielded cylindrical resonator.

represented the experimental and theoretical variations of the resonant frequencies of a dielectric resonator as a function of the distance between the resonator and the ground plane. Experimental and theoretical results agree very well.

VI. Conclusion

The methods allow us to obtain with good accuracy the resonant frequencies of isolated and shielded resonators; this is important for the development of microwave integrated filters using stable dielectric resonators.

Appendix

A. The Cylindrical Resonator

1) *The Isolated Resonator:*

a) *Definition of approximation* 2_H*:* Inside the dielectric sample, $-H/2 < z < +H/2$,

$$E_{z_i} = J_n(k_c r) \sin n\theta\{\chi_1 \cos \beta z + \chi_2 \sin \beta z\}$$
$$H_{z_i} = J_n(k_c r) \cos n\theta\{\chi_3 \cos \beta z + \chi_4 \sin \beta z\}. \tag{18}$$

Outside the dielectric sample, $z < -H/2$ and $z > H/2$,

$$E_{z_o} = \chi_5 J_n(k_c r) \sin n\theta e^{-\alpha z}$$
$$H_{z_o} = \chi_6 J_n(k_c r) \cos n\theta e^{-\alpha z}. \tag{19}$$

χ_i $(i = 1, 2, \ldots, 6)$ are constant values, with α and β as given in relations (3) and (4).

b) *Definition of approximation* 2_a*:* For hybrid modes $EH_{n,m,p}$, inside the rod, $r < a$,

$$E_{z_i} = A_1 J_n(k_{c_i} r) \cos n\theta e^{-j\beta' z}$$
$$H_{z_i} = B_1 J_n(k_{c_i} r) \sin n\theta e^{-j\beta' z} \tag{20}$$

outside the rod, $r > a$,

$$E_{z_o} = C_1 K_n(k_{c_o} r) \cos n\theta e^{-j\beta' z}$$
$$H_{z_o} = D_1 K_n(k_{c_o} r) \sin n\theta e^{-j\beta' z}. \tag{21}$$

$K_n(k_{c_o} r)$ is the modified Bessel's function of the second kind or the hyperbolic Bessel's function of the second kind.

B. The Rectangular Resonator

1) *The Isolated Resonator:*

a) *Definition of approximation* 2_a *for hybrid modes:* Inside the dielectric sample, $-a/2 < x < a/2$,

$$E_{z_i} = E_1 \cos k_{x_1} x \cos k_y y$$
$$H_{z_i} = H_1 \sin k_{x_1} x \sin k_y y \tag{22}$$

outside the dielectric sample, $x < -a/2$ and $x > a/2$,

$$E_{z_o} = E_2 e^{-k_x x} \cos k_y y$$
$$H_{z_o} = H_2 e^{-k_x x} \sin k_y y. \tag{23}$$

b) *Secular equation when the surfaces* $x = \pm a/2$ *and* $z = 0, H$ *are perfect magnetic walls:*

$$\beta'^2 k_x^2 \left(\frac{1}{k_{c_1}^2} + \frac{1}{k_{c_2}^2}\right) k_o^2 \varepsilon_2 \left(\frac{k_{y_2}^2}{k_{c_2}^2} + \frac{k_{y_1} k_{y_2}}{k_{c_1}^2 k_{c_2}^2} \tan k_{y_2} b\right) - k_o^2 \varepsilon_1 \left(\frac{k_{y_1}^2}{k_{c_1}^4} + \frac{k_{y_1} k_{y_2}}{k_{c_1}^2 k_{c_2}^2} \frac{1}{\tan k_{y_1} b}\right) = 0 \tag{24}$$

with

$$k_{y_2}^2 = k_o^2 \varepsilon_2 - (k_x^2 + \beta'^2)$$
$$k_{y_1}^2 = (k_y^2 + \beta'^2) - k_o^2 \varepsilon_1.$$

c) *Definition of approximation* $2_{a,b}$*:* Inside the resonator

$$H_{z_2} = H_2 \sin k_{x_2} x \sin k_{y_2} y$$
$$E_{z_2} = E_2 \cos k_{x_2} x \cos k_{y_2} y. \tag{25}$$

Outside the resonator, $x \geq a/2$ and $y > b$,

$$H_{z_1} = H_1 e^{-k_{x1} x} \sin k_{y_2} y$$
$$E_{z_1} = E_1 e^{-k_{x1} x} \cos k_{y_2} y$$
$$H_{z_3} = H_3 e^{-k_{y3} y} \sin k_{x_2} x$$
$$E_{z_3} = E_3 e^{-k_{y3} y} \cos k_{x_2} x. \tag{26}$$

2) *The Shielded Resonator:*

a) *Definition of approximation* 2_a *for hybrid modes:* Inside the resonator, $-(a/2) < x < (a/2)$,

$$E_{z_2} = E_2 \cos k_{x_2} x \cos k_y y$$
$$H_{z_2} = H_2 \sin k_{x_2} x \sin k_y y. \tag{27}$$

Outside the resonator, $(a/2) < x < (d - 1)$,

$$E_{z_1} = E_1 \sin k_{x_1}(d - x) \cos k_y y$$
$$H_{z_1} = H_1 \cos k_{x_1}(d - x) \sin k_y y. \tag{28}$$

C. Determinant D

$$
D = \begin{vmatrix}
\frac{1}{k_{c_1}^2}\beta k_{y_1}\cos k_{x_1}\frac{a}{2} & \frac{1}{k_{c_1}^2}\cos k_{x_1}\frac{a}{2} & \frac{-1}{k_{c_2}^2}\beta k_{y_1} e^{-k_{x1}a/2} & \frac{1}{k_{c_2}^2}\omega\mu_2 k_{x_2} e^{-k_{x2}a/2} \\
\frac{1}{k_{c_2}^2}\omega\varepsilon_2 k_{x_1}\sin k_{x_1}\frac{a}{2} & -\frac{\beta k_{y_1}}{k_{c_1}^2}\sin k_{x_1}\frac{a}{2} & -\frac{1}{k_{c_2}^2}\omega\varepsilon_2 k_{x_2} e^{-k_{x2}a/2} & \frac{\beta k_{y_1}}{k_{c_2}^2} e^{-k_{x2}a/2} \\
\frac{\beta k_{y_1}}{k_{c_1}^2}\sin k_{y_1} b & \frac{\omega\mu}{k_{c_1}^2}\sin k_{y_1} b & \frac{\beta k_{y_3}}{k_{c_3}^2}\frac{\cos k_{y_1} b}{\cos k_{x_1}\frac{a}{2}} e^{-k_{x2}a/2} & \frac{\omega\mu_3 k_{c_1}}{k_{c_3}^2}\frac{\sin k_{y_1} b}{\sin k_{x_1}\frac{a}{2}} e^{-k_{x2}a/2} \\
\frac{\omega\varepsilon_1}{k_{c_1}^2}\cos k_{y_1} b & \frac{\beta k_{y_1}}{k_{c_1}^2}\cos k_{y_1} b & \frac{\omega\varepsilon_3}{k_{c_3}^2}\frac{\cos k_{y_1} b}{\cos k_{x_1}\frac{a}{2}} e^{-k_{x2}a/2} & -\frac{\beta k_{y_3}}{k_{c_3}^2}\frac{\sin k_{y_1} b}{\sin k_{x_1}\frac{a}{2}} e^{-k_{x2}a/2}
\end{vmatrix}
$$

with

$$k_{c_1}^2 = k_o^2\,\varepsilon_1 - \beta'^2 = k_{x_1}^2 + k_{y_1}^2$$

$$k_{c_2}^2 = k_o^2\,\varepsilon_2 - \beta'^2 = k_{y_1}^2 + k_{y_2}^2$$

$$k_{c_3}^2 = k_o^2\,\varepsilon_3 - \beta'^2 = k_{x_1}^2 + k_{y_3}^2.$$

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